Final Assessment

MTH 403, Semester 1, 2021-22

Duration: 120 minutes

Maximum Points: 50

Directions

- *While writing solutions, please ensure that you sufficiently explain and justify all intermediate arguments leading to any conclusions that you may draw along the way.*
- *Each statement (or argument) in your solution must be clearly explained, and must be devoid of any logical fallacies or gaps.*
- *The fifth question is a bonus question which carries an additional 10 points. You are advised to attempt this question only after you have tried solving all remaining questions.*

Questions

1. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be functions of class C^1. If $c = (x_0, y_0, z_0)$ satisfies both equations $f(x, y, z) = 0$ and $g(x, y, z) = 0$, and $\frac{\partial(f,g)}{\partial(x,y,z)}$ is of rank 2 at c, then show that one can solve for x and y in terms of z near c. [10]

2. Let $A \subset \mathbb{R}^n$ be open; let $f : A \to \mathbb{R}$ be of class C^2. For a rectangle $Q \subset A$ show that [10]

$$\int_Q D_2 D_1 f = \int_Q D_1 D_2 f.$$

3. A tensor $T \in \mathscr{T}^k(V)$ is said to *symmetric* if

$$T(v_1, \ldots, v_k) = T(v_{\sigma(1)}, \ldots, v_{\sigma(k)}),$$

for each $\sigma \in S_k$. Let $\mathscr{T}^k_{sym}(V) \subset \mathscr{T}^k(V)$ be the subspace of all symmetric k-tensors. Consider the linear map $Alt : \mathscr{T}^k(V) \to \mathscr{T}^k(V)$. [10+5]

   (a) Show that $\ker(Alt) = \mathscr{T}^k_{sym}(V)$.

   (b) Compute $\dim(\mathscr{T}^k_{sym}(V))$.

4. Let $\omega \neq 0$ be a k-form on an open set $A \subset \mathbb{R}^n$. [5+10]

   (a) Show that there exists a k-chain c such that $\int_c \omega \neq 0$.

   (b) Use $3(a)$ and the fact that $\partial^2 = 0$ to show that $d^2 = 0$.

5. **(Bonus)** Let $D \subset \mathbb{R}^2$ be an open disk. Two paths $\gamma_1, \gamma_2 : [0, 1] \to D$ with the same initial and end points are said to be *homotopic* if there exists a continuous function $H : [0, 1] \times [0, 1] \to D$ such that $H(x, 0) = \gamma_1(x)$, $H(x, 1) = \gamma_2(x)$, $H(0, t) = \gamma_i(0)$, and $H(1, t) = \gamma_i(1)$. If $\gamma_1, \gamma_2 : [0, 1] \to D$ are homotopic paths, then show that for any closed 1-form ω in D, we have [10]

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega.$$